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Force and couple exerted on a moving electromagnetic dipole

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Abstract. A general approach is used to describe particles possessing dipole moment. By this means, exact expressions for the force and couple exerted on a dipole moving in an electromagnetic field are derived in terms of vectors representing the velocity and orientation of the dipole and their derivatives with respect to time. The formulae reduce to the usual expressions for the force and couple when the dipole is at rest, but are valid for all speeds of the dipole.

1. Introduction

The classical theory of charged particles, as a relativistic theory, has been developed from two constituent theories: relativistic particle mechanics and Maxwell–Lorentz electrodynamics. The latter theory principally contains the Maxwell–Lorentz law of force for a charged particle moving in an electromagnetic field

$$f^{\mu} = -eF^{\mu\nu}V_{\nu}.$$

In any system of interacting particles where it is necessary to determine the forces acting, these two theories may be used to obtain the equations of motion of the system. The method is usually based on an action principle which is derived from knowledge of the forces acting on one particle due to its presence in a field created by another. Thus, with the exception of radiation reaction forces, we are able to determine the motion of the system. Simple modification of the Lagrangian enables us to deduce the equations of motion for particles moving in any given electromagnetic environment, and in the simple case where a charge is acted upon by an external electromagnetic field the problem can often be solved explicitly; for example, in the case where an electron is moving in uniform crossed electric and magnetic fields, according to the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(mV) = e\boldsymbol{E}+\frac{e}{c}\,V\wedge \boldsymbol{B}$$

and the motion is well known to be that of a certain kind of helix.

It is our intention to raise the level of the classical theory slightly by allowing our particles to possess spin angular momentum in addition to linear momentum. This means that they also possess moment and are moving subject to couples as well as forces. Although no particles of the same 'status' as the electron exist in nature, with dipole characteristics worthy of investigation, nevertheless it might be useful to know how such particles should behave in theory, according to the classical theory, and also how spinning particles should behave, when the dipole aspect would predominate.

Partly to this end we have derived three-dimensional vector expressions for the force and couple exerted on a moving dipole. For the most part the results which we have obtained are quite new. Although the expressions we have derived reduce to the usual (vector) expressions for the force and couple when the dipole centre is at rest, the method of their derivation has followed a four-dimensional approach. This is because, unlike the derivation of the Maxwell–Lorentz force formula for a moving charge, the only really successful method for investigating the force on a moving dipole is via the suffix method. We have used this method in the following section in deriving equations of motion for a dipole, which are analogous to the moving charge equations

$$c^2 \frac{\mathrm{d}}{\mathrm{d}\tau}(m_0 V^\mu) = -eF^{\mu\nu} V_\nu$$

and so expressions for the force and couple in four-dimensional suffix notation result directly from them.

The three-dimensional vector expressions for the force and couple which we have derived look fairly simple but we have not applied them to any practical problems. They may be of some use in cases where the use of suffixes is not desirable. A check can be made on the validity of the results we have obtained by taking the case where a dipole moves with uniform velocity. In this case we may transform the usual expressions for the force and couple acting on the dipole in the rest frame by Lorentz transformation to the new frame. In this special case the transformation is not quite as straightforward as one might imagine and the corresponding tensor transformations for the velocity, moment, force and couple have to be written in three-dimensional form to deduce the final result.

It is sometimes useful to consider the fields of slowly moving charges (first-order theory) and so we have derived expressions for the force and couple exerted on a slowly moving dipole. These expressions, for what they may be worth, bear a very close resemblance to the Maxwell–Lorentz force formula for a moving charge. This fact is capable of independent verification.

2. The moving dipole

We consider a moving point in space as represented as a 'four-event' x^μ, where $x^0 \equiv ct$. The x^μ denote the components of a four-vector referred to an inertial frame. Greek suffixes take the values 0, 1, 2, 3, and we shall raise and lower suffixes by means of the metric tensor $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$, where the metric of special relativity is taken in the form $\mathrm{d}\tau^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$. Suffixes prefixed by a comma will denote differentiation with respect to the event x^μ and differentiation with respect to the proper time τ is denoted by a dot. Thus the four-velocity $\dot{z}^\mu$ at an event on a given world line $z^\mu = z^\mu(\tau)$ has the value $V^\mu = (\beta, \beta \boldsymbol{V}/c)$, where $\beta = (1 - V^2/c^2)^{-1/2}$ (since $\dot{z}^\mu \dot{z}_\mu = 1$).

We suppose that the moment of a moving point-dipole (we consider an electric dipole for simplicity) is represented by means of a 'four-moment' $q^\mu = (q, \boldsymbol{q})$ which is defined at the event to be orthogonal (in a four-dimensional sense) to the four-velocity of the dipole centre V^μ. Thus $q^\mu V_\mu = 0$. In order to relate this moment with the ordinary three-dimensional vector moment ($\boldsymbol{n}$), we must consider the antisymmetrical moment tensor (or see, for example, Panofsky and Phillips 1964, p. 438), or six-vector

$$p^{\mu\nu} = q^\mu V^\nu - q^\nu V^\mu \tag{2.1}$$

whose components are given by

$$(p^{01}, p^{02}, p^{03}) \equiv -\beta\left(\boldsymbol{q} - \frac{q\boldsymbol{V}}{c}\right)$$

$$(p^{23}, p^{31}, p^{12}) \equiv \beta\left(\boldsymbol{q} - \frac{q\boldsymbol{V}}{c}\right) \wedge \frac{\boldsymbol{V}}{c}\,. \tag{2.2}$$

Writing

$$n = q - \frac{qV}{c} \tag{2.3}$$

it can be verified that the latter quantity represents the ordinary three-dimensional vector moment of the dipole.† This fact is not surprising in view of the covariance of the quantities (2.2) from the values in an instantaneous rest frame. It follows that any formal tensor expressions containing (2.1) may be reduced quite readily to three-dimensional vector quantities by taking components.

In order to describe the effect of Lorentz contraction on the moving moment we shall now consider the case of a dipole of constant strength ($(q^\mu q_\mu)^{1/2}$ = constant), although as it happens the following formula remains valid in general. The formula we quote is

$$\frac{|\boldsymbol{n}|}{M} = \left(\frac{1 - V^2/c^2}{1 - |\hat{\boldsymbol{n}} \wedge \boldsymbol{V}|^2/c^2}\right)^{1/2} \tag{2.4}$$

where M is the constant strength-value of the moment measured in the dipole's rest frame and $|\boldsymbol{n}|$ is the magnitude of the observed moment. An observer who is moving with the dipole always records the constant value M for the magnitude of the moment. On the other hand, to an observer with respect to whom the dipole is moving, the infinitesimal distance between the poles of the dipole, and therefore the measured moment $|\boldsymbol{n}|$, will differ from their values in the instantaneous rest frame of the dipole, by the above ratio. We can illustrate the validity of this formula in two cases where the dipole moves in a direction (i) perpendicular to, and (ii) parallel to, its axis. For (i) the formula gives no change and for (ii) the formula gives a pure Lorentz contraction, as we should expect. The justification for (2.4) for all other cases may be made on the basis of a dipole as an 'infinitesimal rigid rod' according to the requirements of Lorentz invariance. (A derivation is given in an appendix.)

We hope we may be forgiven if we now venture off, slightly at a tangent, into some tensor algebra. The reason has been given in the introduction, and we need to set up a working model of a moving dipole.

It is necessary to state that we shall need the alternating symbols $\xi_{\mu\nu\alpha\beta}$, $\xi^{\mu\nu\alpha\beta}$ which are relative tensors having the numerical value of $\delta^{0123}_{\mu\nu\alpha\beta}$ (i.e. they are equal to $+1$ if μ, ν, α, β is an even permutation of 0, 1, 2, 3; -1 if μ, ν, α, β is an odd permutation of 0, 1, 2, 3, and zero otherwise). Duality on a skew pair of indices will be defined according to the definitions

$$\mathrm{i}F^*_{\mu\nu} = \tfrac{1}{2}g^{1/2}\xi_{\mu\nu\alpha\beta}F^{\alpha\beta} \qquad (\text{or } \mathrm{i}F^{*\mu\nu} = \tfrac{1}{2}g^{-1/2}\xi^{\mu\nu\alpha\beta}F_{\alpha\beta})$$

where $g = \det(g_{\mu\nu}) = -1$. The ξ symbols are therefore related to the familiar ϵ tensor by the following relations:

$$\epsilon_{\mu\nu\alpha\beta} = \mathrm{i}g^{1/2}\xi_{\mu\nu\alpha\beta}, \qquad \epsilon^{\mu\nu\alpha\beta} = \mathrm{i}g^{-1/2}\xi^{\mu\nu\alpha\beta}.$$

Effectively the ξ symbol changes sign when its indices are raised or lowered (unlike the ϵ tensor). The following notation will be used for the electromagnetic field *in vacuo* (Gaussian units):

$$(F^{01}, F^{02}, F^{03}) \equiv \boldsymbol{E}, \qquad (F^{23}, F^{31}, F^{12}) \equiv \boldsymbol{B}$$

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}, \qquad A^\mu \equiv (\phi, \boldsymbol{A}).$$

† In view of orthogonality, q and $\boldsymbol{q}$ are related by the equation $q = \boldsymbol{q} \,.\, \boldsymbol{V}/c$. From this the physical interpretation of $\boldsymbol{n}$ becomes clear (Ellis 1963).

Although it is well known that the invariant interaction Lagrangian for a charge

$$\mathrm{L}_{\mathrm{int}} = \beta\left(e\phi - \frac{e}{c}\boldsymbol{A}\,.\,\boldsymbol{V}\right) = eV^{\alpha}A_{\alpha}$$

yields the generalized momentum of the form

$$p^{\mu} = \frac{\partial \mathrm{L}}{\partial V_{\mu}} = G^{\mu} + eA^{\mu}$$

where G^{μ} is the mechanical momentum, for a dipole the corresponding situation is not often referred to. The corresponding Lagrangian is $-\frac{1}{2}p^{\alpha\beta}F_{\alpha\beta}$, and this gives rise to an additional term in the generalized momentum, arising from differentiation of the moment tensor $p^{\alpha\beta}$ with respect to V_{μ}. A straightforward calculation will show that

$$p^{\mu} = \frac{\partial \mathrm{L}}{\partial V_{\mu}} = G^{\mu} + eA^{\mu} + q_{\alpha}F^{\mu\alpha}. \tag{2.5}$$

It is our purpose to use expression (2.5) to work out the equations of motion of a pole–plus-dipole particle, following a general approach such as the one given by Barut (1964). The argument follows the method on page 77 of his book. The Lagrangian we consider is a function of

$$A_{\mu}(x),\ A_{\mu,\nu}(x),\ \dot{x}_{\mu},\ q^{\mu}{}_{(\alpha)},\ \dot{q}^{\mu}{}_{(\alpha)}$$

where x^{μ} are 'global' coordinates (referring to the motion of the centre of the pole–plus-dipole particle) and the $q^{\mu}{}_{(\alpha)}$ are internal space coordinates, or 'degrees of freedom,' corresponding to the spin ($\alpha = 1, 2, 3$), orthogonal to the four-velocity. (We have here excluded the vector corresponding to $\alpha = 0$ since we assume a Lagrangian independent of $\ddot{x}^{\mu}$.) The latter coordinates are introduced principally in order to refer to the spin angular momentum of the particle. Variation of such a Lagrangian leads to the following expression:

$$\begin{aligned}\delta \mathrm{L} &= \frac{\partial \mathrm{L}}{\partial \dot{x}^{\mu}}\delta\dot{x}^{\mu} + \frac{\partial \mathrm{L}}{\partial q^{\mu}{}_{(\alpha)}}\delta q^{\mu}{}_{(\alpha)} + \frac{\partial \mathrm{L}}{\partial \dot{q}^{\mu}{}_{(\alpha)}}\delta\dot{q}^{\mu}{}_{(\alpha)} + \frac{\partial \mathrm{L}}{\partial A_{\lambda}}\delta A_{\lambda} + \frac{\partial \mathrm{L}}{\partial A_{\lambda,\nu}}\delta A_{\lambda,\nu}\\
&= \left\{\frac{\partial \mathrm{L}}{\partial x^{\mu}} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathrm{L}}{\partial \dot{x}^{\mu}}\right)\right\}\delta x^{\mu} + \left\{\frac{\partial \mathrm{L}}{\partial q^{\mu}{}_{(\alpha)}} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathrm{L}}{\partial \dot{q}^{\mu}{}_{(\alpha)}}\right)\right\}\delta q^{\mu}{}_{(\alpha)}\\
&\quad + \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\left(\frac{\partial \mathrm{L}}{\partial \dot{x}^{\mu}}\right)\delta x^{\mu} + \left(\frac{\partial \mathrm{L}}{\partial \dot{q}^{\mu}{}_{(\alpha)}}\right)\delta q^{\mu}{}_{(\alpha)}\right\} + \frac{\partial \mathrm{L}}{\partial A_{\lambda}}(\delta A_{\lambda} - A_{\lambda,\mu}\delta x^{\mu})\\
&\quad + \frac{\partial \mathrm{L}}{\partial A_{\lambda,\nu}}(\delta A_{\lambda,\nu} - A_{\lambda,\nu\mu}\delta x^{\mu}).\end{aligned}$$

The first two terms enclosed in braces vanish because they are the Euler–Lagrange equations. In the remaining terms we consider the variations to be caused (i) by an infinitesimal translation and (ii) by an infinitesimal Lorentz rotation. The two resulting equations of motion will be for the momentum and angular momentum.

We assume that A_μ and $A_{\mu,\nu}$ enter the Lagrangian explicitly only via interaction (the last antisymmetrically in μ and ν), so that

$$\frac{\partial L}{\partial A_\lambda} = eV^\lambda$$

$$\frac{\partial L}{\partial A_{\lambda,\nu}} = -p^{\lambda\nu} \qquad (= q^\nu V^\lambda - q^\lambda V^\nu).$$

The generalized momentum p^μ is given by

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}$$

and the generalized spin angular momentum we call $S^{\mu\nu}$

$$\begin{aligned} S_{\mu\nu} &= 2\frac{\partial L}{\partial \dot{q}^{[\mu}{}_{(\alpha)}} q_{\nu](\alpha)} \\ &= \frac{\partial L}{\partial \dot{q}^{\mu}{}_{(\alpha)}} q_{\nu(\alpha)} - \frac{\partial L}{\partial \dot{q}^{\nu}{}_{(\alpha)}} q_{\mu(\alpha)}. \end{aligned}$$

For an infinitesimal translation (i)

$$\delta x^\mu = \epsilon^\mu, \qquad \frac{d}{d\tau}(\delta x^\mu) = 0, \qquad \delta A_\mu = \delta q^\mu{}_{(\alpha)} = 0$$

we obtain

$$\begin{aligned} \dot{p}_\mu &= \frac{\partial L}{\partial A_\lambda} A_{\lambda,\mu} + \frac{\partial L}{\partial A_{\lambda,\nu}} A_{\lambda,\nu\mu} \\ &= eA_{\lambda,\mu} V^\lambda - A_{\lambda,\rho\mu} p^{\lambda\rho}. \end{aligned} \tag{2.6}$$

For an infinitesimal rotation (ii),

$$\delta x^\mu = \omega^{\mu\nu} x_\nu, \qquad \delta A_\lambda = \omega_{\lambda\nu} A^\nu, \qquad \delta q^\mu{}_{(\alpha)} = \omega^{\mu\nu} q_{\nu(\alpha)}$$

$$\delta A_{\lambda,\nu} = \omega_{\lambda\rho} A^\rho{}_{,\nu} + \omega_{\nu\rho} A_{\lambda,}{}^\rho, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

we obtain

$$\begin{aligned} &\frac{d}{d\tau}(2p_{[\mu}x_{\nu]} + S_{\mu\nu}) + 2\frac{\partial L}{\partial A^{[\mu}} A_{\nu]} - 2\frac{\partial L}{\partial A_\lambda} A_{\lambda,[\mu} x_{\nu]} \\ &\quad + 2\frac{\partial L}{\partial A^{[\mu}{}_{,\rho}} A_{\nu],\rho} + 2\frac{\partial L}{\partial A_{\lambda,}{}^{[\mu}} A_{\lambda,\nu]} - 2\frac{\partial L}{\partial A_{\lambda,\rho}} A_{\lambda,\rho[\mu} x_{\nu]} = 0 \end{aligned}$$

which reduces with the help of (2.6) to

$$\dot{S}_{\mu\nu} + 2p_{[\mu}V_{\nu]} = -2eV_{[\mu}A_{\nu]} - 2q^\rho V_{[\mu}F_{\nu]\rho} + 2q_{[\mu}V^\rho F_{\nu]\rho}. \tag{2.7}$$

Use of our expression (2.5) for the generalized momentum leads to two equations

(Ellis 1966) from (2.6) and (2.7)

$$\dot{G}_\mu = -eF_{\mu\nu}V^\nu - \dot{q}^\alpha F_{\mu\alpha} - q^\alpha F_{\mu\nu,\alpha}V^\nu \tag{2.8}$$

$$\dot{S}_{\mu\nu} + 2G_{[\mu}V_{\nu]} = 2q_{[\mu}F_{\nu]\alpha}V^\alpha \tag{2.9}$$

which are the required equations of motion of an electric dipole of moment q^μ (with added charge e). It should be pointed out that our internal coordinates $q^\mu{}_{(\alpha)}$ have disappeared from these equations, but they are present implicitly in the value of $S_{\mu\nu}$. Multiplication of the second equation (2.9) by V^ν gives the mechanical momentum

$$G^\mu = m_0c^2V^\mu - \dot{S}^{\mu\nu}V_\nu \tag{2.10}$$

where $m_0c^2 = G^\mu V_\mu$. It is noteworthy that the mechanical momentum is independent of the electromagnetic field, a state of affairs which we would consider desirable. The rest mass of the dipole is m_0, and this may or may not remain constant throughout the motion.

The expression for the four-force on the dipole is clearly given by the second and third terms in the expression for $\dot{G}^\mu$ in (2.8), while the expression for the couple is given by the right-hand side of (2.9). We can make this more recognizable as a couple by expressing the six components of $\dot{S}^{\mu\nu}$ in terms of two four-vectors U^μ, T^μ orthogonal to V^μ

$$-\dot{S}^{\mu\nu} = \delta^{\mu\nu}_{\alpha\beta}U^\alpha V^\beta + \xi^{\mu\nu\alpha\beta}T_\alpha V_\beta. \tag{2.11}$$

(Any bivector may be so represented.) In the instantaneous rest frame, S^{23}, S^{31}, S^{12} represent the negative of the angular momentum and T^1, T^2, T^3 the components of the couple. We can demonstrate that the four-vector U^μ does not form part of the equations of motion by substituting (2.11) into (2.9). We find for the left-hand side of (2.9), calling this $C_{\mu\nu}$,

$$\begin{aligned} C_{\mu\nu} &= \dot{S}_{\mu\nu} - 2\dot{S}_{[\mu\alpha}V^\alpha V_{\nu]} \qquad \text{(from (2.10))} \\ &= \xi_{\mu\nu\alpha\beta}T^\alpha V^\beta + 2\xi_{[\mu\alpha\beta\gamma}T^\beta V^\gamma V^\alpha V_{\nu]} - U_\mu V_\nu + U_\nu V_\mu + 2U_{[\mu}V_\alpha V^\alpha V_{\nu]} - 2U_\alpha V_{[\mu}V^\alpha V_{\nu]} \\ &= \xi_{\mu\nu\alpha\beta}T^\alpha V^\beta. \end{aligned}$$

By an argument similar to the interpretation of the components of the antisymmetric moment tensor (2.2), the components of the observed couple are given by

$$(C_{23}, C_{31}, C_{12}) \equiv -\beta\mathbf{C},\ (C_{01}, C_{02}, C_{03}) \equiv \beta\mathbf{C} \wedge \frac{\mathbf{V}}{c} \tag{2.12}$$

where

$$C_{\mu\nu} = 2q_{[\mu}F_{\nu]\alpha}V^\alpha. \tag{2.13}$$

3. Vector description

Since the vector moment $\mathbf{n}$ is given by

$$\mathbf{n} = \mathbf{q} - \frac{q\mathbf{V}}{c}$$

(expression (2.3)), it follows from the orthogonality condition $q = \boldsymbol{q} \cdot \boldsymbol{V}/c$ that

$$q = \left(\boldsymbol{n} + \frac{q\boldsymbol{V}}{c}\right) \cdot \frac{\boldsymbol{V}}{c}.$$

Hence

$$q = \beta^2 \frac{\boldsymbol{n} \cdot \boldsymbol{V}}{c}$$

and

$$\boldsymbol{q} = \boldsymbol{n} + \beta^2 \left(\frac{\boldsymbol{n} \cdot \boldsymbol{V}}{c}\right) \frac{\boldsymbol{V}}{c}.$$

We therefore have the following expression for q^μ:

$$q^\mu \equiv \beta^2 \left[\frac{\boldsymbol{n} \cdot \boldsymbol{V}}{c}, \quad \boldsymbol{n} - \left\{\frac{\boldsymbol{V}}{c} \wedge \left(\boldsymbol{n} \wedge \frac{\boldsymbol{V}}{c}\right)\right\}\right].$$

Denoting differentiation with respect to ct by primes, we have the following formulae for $\dot{q}$ and $\dot{\boldsymbol{q}}$:

$$\dot{q} = \beta q', \qquad \dot{\boldsymbol{q}} = \beta \boldsymbol{q}'.$$

Hence

$$\dot{q}^\mu \equiv \beta \left[\left(\frac{\beta^2 \boldsymbol{n} \cdot \boldsymbol{V}}{c}\right)', \quad \boldsymbol{n}' + \left\{\left(\frac{\beta^2 \boldsymbol{n} \cdot \boldsymbol{V}}{c}\right) \frac{\boldsymbol{V}}{c}\right\}'\right].$$

Thus we can write the dipole four-force $\dot{G}^\mu$ from (2.8) in terms of three-dimensional vectors as follows:

$$\begin{aligned} \dot{G}^\mu &\equiv (P, \boldsymbol{F}) = -\dot{q}_\alpha F^{\mu\alpha} - q^\alpha F^{\mu\nu}{}_{,\alpha} V_\nu \\ &\equiv \Bigg[\beta \boldsymbol{E} \cdot \boldsymbol{q}' + q\boldsymbol{E}' \cdot \frac{\beta \boldsymbol{V}}{c} + \frac{\beta \boldsymbol{V}}{c} \cdot \{(\boldsymbol{q} \cdot \nabla)\boldsymbol{E}\}, \quad \beta \boldsymbol{q}' \wedge \boldsymbol{B} + \beta q' \boldsymbol{E} + q\beta \boldsymbol{E}' + q \frac{\beta \boldsymbol{V}}{c} \wedge \boldsymbol{B}' \\ &\quad + \beta(\boldsymbol{q} \cdot \nabla)\boldsymbol{E} + \frac{\beta \boldsymbol{V}}{c} \wedge \{(\boldsymbol{q} \cdot \nabla)\boldsymbol{B}\}\Bigg] \end{aligned}$$

where a prime acting on a field quantity is understood as a partial derivative. The components of the four-force then become

$$\begin{aligned} P &= \beta \boldsymbol{n}' \cdot \boldsymbol{E} + \beta \left\{\beta^2 \left(\frac{\boldsymbol{n} \cdot \boldsymbol{V}}{c}\right)\left(\frac{\boldsymbol{E} \cdot \boldsymbol{V}}{c}\right)\right\}' \\ &\quad + \beta^3 \left[\left\{\boldsymbol{n} + \frac{\boldsymbol{V}}{c} \wedge \left(\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{n}\right)\right\} \cdot \nabla\right] \left(\frac{\boldsymbol{E} \cdot \boldsymbol{V}}{c}\right) \\ &\left(= \frac{\boldsymbol{F} \cdot \boldsymbol{V}}{c} + \beta \left(\boldsymbol{E} + \frac{\boldsymbol{V}}{c} \wedge \boldsymbol{B}\right) \cdot \left\{\boldsymbol{n}' + \beta^2 \left(\frac{\boldsymbol{n} \cdot \boldsymbol{V}}{c}\right) \frac{\boldsymbol{V}'}{c}\right\}\right) \end{aligned} \tag{3.1}$$

$$\begin{aligned} \boldsymbol{F} &= \beta \boldsymbol{n}' \wedge \boldsymbol{B} + \beta \left\{\beta^2 \frac{\boldsymbol{n} \cdot \boldsymbol{V}}{c} \left(\boldsymbol{E} + \frac{\boldsymbol{V}}{c} \wedge \boldsymbol{B}\right)\right\}' \\ &\quad + \beta^3 \left[\left\{\boldsymbol{n} + \frac{\boldsymbol{V}}{c} \wedge \left(\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{n}\right)\right\} \cdot \nabla\right] \left(\boldsymbol{E} + \frac{\boldsymbol{V}}{c} \wedge \boldsymbol{B}\right) \end{aligned} \tag{3.2}$$

where ∇ is taken to act on $\boldsymbol{E}$ and $\boldsymbol{B}$ only. (Note that a prime is taken to act on all quantities.) These expressions simplify considerably in the case of slowly varying and especially static fields. For example, in the case of a static $\boldsymbol{B}$ field the expression for the force reduces to

$$\boldsymbol{F} = \beta \boldsymbol{n}' \wedge \boldsymbol{B} + \beta\left\{\beta^2 \left(\frac{\boldsymbol{n} \cdot \boldsymbol{V}}{c}\right) \frac{\boldsymbol{V}}{c}\right\}' \wedge \boldsymbol{B}$$

$$\left(= \beta\left[\beta^2\left\{\boldsymbol{n} + \frac{\boldsymbol{V}}{c} \wedge \left(\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{n}\right)\right\}\right]' \wedge \boldsymbol{B}\right).$$

In a similar way to the evaluation of $\boldsymbol{F}$ and P, the couple from the derived expressions (2.12) and (2.13) becomes

$$\boldsymbol{C} = \boldsymbol{q} \wedge \boldsymbol{E} + (\boldsymbol{q} \cdot \boldsymbol{B})\frac{\boldsymbol{V}}{c} - \left(\frac{\boldsymbol{q} \cdot \boldsymbol{V}}{c}\right)\boldsymbol{B}$$

$$= \beta^2\left\{\boldsymbol{n} + \frac{\boldsymbol{V}}{c} \wedge \left(\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{n}\right)\right\} \wedge \left(\boldsymbol{E} + \frac{\boldsymbol{V}}{c} \wedge \boldsymbol{B}\right). \qquad (3.3)$$

In many practical cases we could neglect all terms where c occurs more than twice in the denominator. If the fields do not vary wildly and speeds are not very high, the following formulae will be useful, although their exactness is not retained:

$$P = \boldsymbol{n}' \cdot \boldsymbol{E} + (\boldsymbol{n} \cdot \nabla)\left(\frac{\boldsymbol{E} \cdot \boldsymbol{V}}{c}\right)$$

$$\boldsymbol{F} = \boldsymbol{n}' \wedge \boldsymbol{B} + \left\{\left(\frac{\boldsymbol{n} \cdot \boldsymbol{V}}{c}\right)\boldsymbol{E}\right\}' + \beta^3(\boldsymbol{n} \cdot \nabla)\boldsymbol{E} + (\boldsymbol{n} \cdot \nabla)\left(\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{B}\right) + \left[\left\{\frac{\boldsymbol{V}}{c} \wedge \left(\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{n}\right)\right\} \cdot \nabla\right]\boldsymbol{E}$$

$$\boldsymbol{C} = \beta^2 \boldsymbol{n} \wedge \boldsymbol{E} + \boldsymbol{n} \wedge \left(\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{B}\right) + \left\{\frac{\boldsymbol{V}}{c} \wedge \left(\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{n}\right)\right\} \wedge \boldsymbol{E}.$$

If we go further and neglect all terms where c occurs more than once in the denominator, we obtain the following rough expressions for P, $\boldsymbol{F}$ and $\boldsymbol{C}$, which still show relativistic aberrations from the usual static formulae. For this case, the effects of Lorentz contraction (formula (2.4)) are ignored:

$$P = \boldsymbol{n}' \cdot \boldsymbol{E} + \frac{\boldsymbol{F} \cdot \boldsymbol{V}}{c}$$

$$\boldsymbol{F} = \boldsymbol{n}' \wedge \boldsymbol{B} + (\boldsymbol{n} \cdot \nabla)\left(\boldsymbol{E} + \frac{\boldsymbol{V}}{c} \wedge \boldsymbol{B}\right)$$

$$\boldsymbol{C} = \boldsymbol{n} \wedge \left(\boldsymbol{E} + \frac{\boldsymbol{V}}{c} \wedge \boldsymbol{B}\right).$$

The formulae are good approximations for slow speeds and when the fields are slowly varying with time. They bear an obvious resemblance to the Lorentz force formula for a moving charge, from which they may be deduced. They are 'intuitive' extensions of the static formulae (the first arises by energy considerations, and the second and third by the replacement of $\boldsymbol{E}$ by $\boldsymbol{E} + \boldsymbol{V} \wedge \boldsymbol{B}/c$).

4. The case of constant velocity

When the dipole centre is at rest, the quantities P, F and C reduce to well-known ones:

$$P_{(0)} = \dot{n}_{(0)} \cdot E_{(0)} \tag{4.1}$$

$$F_{(0)} = \dot{n}_{(0)} \wedge B_{(0)} + (n_{(0)} \cdot \nabla_{(0)}) E_{(0)} \tag{4.2}$$

$$C_{(0)} = n_{(0)} \wedge E_{(0)}. \tag{4.3}$$

We may verify that, if the velocity V is constant, the expressions (3.1), (3.2), (3.3) arise from these by Lorentz transformation with velocity $-V$. Taking (4.1) and (4.2) first, with the aid of the Lorentz 'matrix'

$$\mathsf{L}_V = \begin{pmatrix} \beta & -\dfrac{\beta V}{c} \\ -\dfrac{\beta V}{c} & \delta_{ij} + (\beta - 1)\dfrac{V_i V_j}{V^2} \end{pmatrix}$$

we have, by transformation,

$$\begin{pmatrix} P \\ F \end{pmatrix} = \mathsf{L}_{-V} \begin{pmatrix} P_{(0)} \\ F_{(0)} \end{pmatrix}.$$

Hence

$$P = \beta P_{(0)} + \left(\frac{\beta V}{c}\right) \cdot F_{(0)}$$

$$F = \left(\frac{\beta V}{c}\right) P_{(0)} + F_{(0)} + (\beta - 1)\left(\frac{V \cdot F_{(0)}}{V^2}\right) V. \tag{4.4}$$

These expressions still contain $E_{(0)}$, $B_{(0)}$, $\nabla_{(0)}$ and $n_{(0)}$. $E_{(0)}$ and $B_{(0)}$ can be eliminated in terms of E and B by the following:

$$\left.\begin{aligned} E_{(0)} &= \beta E + \frac{\beta V}{c} \wedge B + (1 - \beta)\left(\frac{V \cdot E}{V^2}\right) V \\ B_{(0)} &= \beta B - \frac{\beta V}{c} \wedge E + (1 - \beta)\left(\frac{V \cdot B}{V^2}\right) V \end{aligned}\right\} \tag{4.5}$$

which arise by transforming the ordinary electromagnetic field 'matrix' $\mathscr{F}$: $\mathscr{F}_{(0)} = \mathsf{L}_V \mathscr{F} \mathsf{L}_V{}^T$ (or see, for example, Panofsky and Phillips 1964, p. 330). The operator $\nabla_{(0)}$ may be eliminated using

$$\nabla_{(0)} = \frac{\beta V}{c} \frac{\partial}{\partial ct} + \nabla + (\beta - 1)\frac{V}{V^2}(V \cdot \nabla) \tag{4.6}$$

which is obtained from

$$\begin{pmatrix} \dfrac{\partial}{\partial ct_{(0)}} \\ -\nabla_{(0)} \end{pmatrix} = \mathsf{L}_V \begin{pmatrix} \dfrac{\partial}{\partial ct} \\ -\nabla \end{pmatrix}.$$

Finally, the expression for $n_{(0)}$ may be deduced by Lorentz-transforming the moment six-vector $(-\beta n, \beta n \wedge V/c)$ of (2.2) to its value $(-n_{(0)}, \mathbf{0})$ in the rest frame. This can be done most easily by substituting $-\beta \boldsymbol{n}$ and $-\boldsymbol{n}_{(0)}$ for $\boldsymbol{E}$ and $\boldsymbol{E}_{(0)}$ in (4.5):

$$\begin{aligned}
\boldsymbol{n}_{(0)} &= \beta(\beta \boldsymbol{n}) + \frac{\beta \boldsymbol{V}}{c} \wedge \left(\frac{\boldsymbol{V}}{c} \wedge \beta \boldsymbol{n}\right) + (1-\beta)\left(\frac{\boldsymbol{V} \cdot \beta \boldsymbol{n}}{V^2}\right) \boldsymbol{V} \\
&= \boldsymbol{n} + \beta^2 \left(\frac{\boldsymbol{V}}{c} \cdot \boldsymbol{n}\right) \frac{\boldsymbol{V}}{c} + \beta(1-\beta)\left(\frac{\boldsymbol{V} \cdot \boldsymbol{n}}{V^2}\right) \boldsymbol{V} \\
&= \boldsymbol{n} + \beta \left\{\frac{\beta V^2}{c^2} + (1-\beta)\right\}\left(\frac{\boldsymbol{V} \cdot \boldsymbol{n}}{V^2}\right) \boldsymbol{V} \\
&= \boldsymbol{n} + (\beta - 1)\left(\frac{\boldsymbol{V} \cdot \boldsymbol{n}}{V^2}\right) \boldsymbol{V} \\
&\left(= \beta \boldsymbol{n} + (\beta - 1)\frac{\boldsymbol{V} \wedge (\boldsymbol{V} \wedge \boldsymbol{n})}{V^2}\right).
\end{aligned} \tag{4.7}$$

Thus, finally, substituting (4.5), (4.6) and (4.7) into (4.4), we should obtain the original expressions for P and $\boldsymbol{F}$ ((3.1) and (3.2)). Since this calculation is very lengthy we may show the result in the case of static fields. From (4.1), (4.2) and (4.4) we have

$$\begin{aligned}
P &= \beta \dot{\boldsymbol{n}}_{(0)} \cdot \boldsymbol{E}_{(0)} + \frac{\beta \boldsymbol{V}}{c} \cdot \dot{\boldsymbol{n}}_{(0)} \wedge \boldsymbol{B}_{(0)} \\
&= \left[\beta^2\left(\boldsymbol{E} + \frac{\boldsymbol{V}}{c} \wedge \boldsymbol{B}\right) \cdot \dot{\boldsymbol{n}} + \beta(\beta-1)\left(\frac{\boldsymbol{V} \cdot \boldsymbol{n}}{V^2}\right)(\boldsymbol{V} \cdot \boldsymbol{E})\{\beta - 1 - (\beta - 1)\}\right] \\
&\quad + \left\{\frac{\beta^2 \boldsymbol{V}}{c} \cdot \dot{\boldsymbol{n}} \wedge \left(\boldsymbol{B} - \frac{\boldsymbol{V}}{c} \wedge \boldsymbol{E}\right)\right\} \\
&= \left(\beta^2 - \frac{\beta^2 V^2}{c^2}\right)(\dot{\boldsymbol{n}} \cdot \boldsymbol{E}) + \beta^2\left(\frac{\dot{\boldsymbol{n}} \cdot \boldsymbol{V}}{c}\right)\left(\frac{\boldsymbol{E} \cdot \boldsymbol{V}}{c}\right) \\
&= \beta \boldsymbol{n}' \cdot \boldsymbol{E} + \beta^3\left(\frac{\boldsymbol{n}' \cdot \boldsymbol{V}}{c}\right)\left(\frac{\boldsymbol{E} \cdot \boldsymbol{V}}{c}\right)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{F} &= \beta(\dot{\boldsymbol{n}}_{(0)} \cdot \boldsymbol{E}_{(0)}) \frac{\boldsymbol{V}}{c} + \dot{\boldsymbol{n}}_{(0)} \wedge \boldsymbol{B}_{(0)} + (\beta - 1)\left(\frac{\boldsymbol{V} \cdot \dot{\boldsymbol{n}}_{(0)} \wedge \boldsymbol{B}_{(0)}}{V^2}\right) \boldsymbol{V} \\
&= \left[\beta^2\left\{\left(\boldsymbol{E} + \frac{\boldsymbol{V}}{c} \wedge \boldsymbol{B}\right) \cdot \dot{\boldsymbol{n}}\right\}\frac{\boldsymbol{V}}{c}\right] \\
&\quad + \left[\beta \dot{\boldsymbol{n}} \wedge \left(\boldsymbol{B} - \frac{\boldsymbol{V}}{c} \wedge \boldsymbol{E}\right) + (1-\beta)\left(\frac{\boldsymbol{V} \cdot \boldsymbol{B}}{V^2}\right)\dot{\boldsymbol{n}} \wedge \boldsymbol{V}\right. \\
&\quad \left. + \beta(1-\beta)\left(\frac{\boldsymbol{V} \cdot \dot{\boldsymbol{n}}}{V^2}\right)\left\{\left(\boldsymbol{B} - \frac{\boldsymbol{V}}{c} \wedge \boldsymbol{E}\right) \wedge \boldsymbol{V}\right\}\right] \\
&\quad + \left[-\beta(1-\beta)\left\{\frac{\boldsymbol{V} \cdot \dot{\boldsymbol{n}} \wedge (\boldsymbol{B} - \boldsymbol{V} \wedge \boldsymbol{E}/c)}{V^2}\right\}\boldsymbol{V}\right]
\end{aligned}$$

$$= \beta^2\left(\frac{\dot{\boldsymbol{n}} \cdot \boldsymbol{V}}{c}\right)\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right)+\beta^2\dot{\boldsymbol{n}}\wedge\left\{\frac{\boldsymbol{V}}{c}\wedge\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right)\right\}$$
$$+\dot{\boldsymbol{n}}\wedge\left\{\beta\left(\boldsymbol{B}-\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{E}\right)+(1-\beta)\left(\frac{\boldsymbol{V} \cdot \boldsymbol{B}}{V^2}\right)\boldsymbol{V}\right\}$$
$$+\left\{\frac{\beta(1-\beta)}{V^2}\right\}\dot{\boldsymbol{n}}\wedge\left[\boldsymbol{V}\wedge\left\{\boldsymbol{V}\wedge\left(\boldsymbol{B}-\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{E}\right)\right\}\right]$$
$$= \beta^2\left(\frac{\dot{\boldsymbol{n}} \cdot \boldsymbol{V}}{c}\right)\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right)+\left\{-\frac{\beta^2V^2}{c^2}+\beta-\beta(1-\beta)\right\}\dot{\boldsymbol{n}}\wedge\boldsymbol{B}$$
$$+\{\beta^2-\beta+\beta(1-\beta)\}\dot{\boldsymbol{n}}\wedge\left(\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{E}\right)$$
$$+\left\{\beta^2+(1-\beta)\frac{c^2}{V^2}+\beta(1-\beta)\frac{c^2}{V^2}\right\}\left(\frac{\boldsymbol{V} \cdot \boldsymbol{B}}{c}\right)\frac{\dot{\boldsymbol{n}}\wedge\boldsymbol{V}}{c}$$
$$= \beta\boldsymbol{n}'\wedge\boldsymbol{B}+\beta^3\left(\frac{\boldsymbol{n}' \cdot \boldsymbol{V}}{c}\right)\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right).$$

This verifies the expression.

Similarly, since the couple $\boldsymbol{C}$ transforms as a six-vector, we may deduce that the expression for the couple (3.3) is obtained from its rest value (4.3) by carrying out a Lorentz transformation with velocity $-\boldsymbol{V}$ of the couple six-vector $(-\beta\boldsymbol{C}\wedge\boldsymbol{V}/c,\ -\beta\boldsymbol{C})$ to its value $(\boldsymbol{0},\ -\boldsymbol{C}_{(0)})$ in the rest frame. For this we may use the second of the expressions (4.5) with $\boldsymbol{V}$ replaced by $-\boldsymbol{V}$, $\boldsymbol{B}_{(0)}$ by $-\beta\boldsymbol{C}$, $\boldsymbol{B}$ by $-\boldsymbol{C}_{(0)}$ and $\boldsymbol{E}$ by $\boldsymbol{0}$:

$$-\beta\boldsymbol{C} = \beta(-\boldsymbol{C}_{(0)})+(1-\beta)\left(-\frac{\boldsymbol{V} \cdot \boldsymbol{C}_{(0)}}{V^2}\right)\boldsymbol{V}.$$

Hence

$$\boldsymbol{C} = \boldsymbol{C}_{(0)}+\left(\frac{1-\beta}{\beta}\right)\left(\frac{\boldsymbol{V} \cdot \boldsymbol{C}_{(0)}}{V^2}\right)\boldsymbol{V}$$
$$= \beta\boldsymbol{n}\wedge\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right)+(1-\beta)\left(\frac{\boldsymbol{V} \cdot \boldsymbol{E}}{V^2}\right)\boldsymbol{n}\wedge\boldsymbol{V}$$
$$-\beta(1-\beta)\frac{\boldsymbol{V} \cdot \boldsymbol{n}}{V^2}\left\{\boldsymbol{V}\wedge\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right)\right\}$$
$$+(1-\beta)\left\{\boldsymbol{V} \cdot \boldsymbol{n}\wedge\left(\frac{\boldsymbol{E}+\boldsymbol{V}\wedge\boldsymbol{B}/c}{V^2}\right)\right\}\boldsymbol{V}$$
$$= \boldsymbol{n}\wedge\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right)+\left(\frac{1-\beta}{V^2}\right)\boldsymbol{n}\wedge\left[\boldsymbol{V}\wedge\left\{\boldsymbol{V}\wedge\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right)\right\}\right]$$
$$-\beta(1-\beta)\frac{\boldsymbol{V} \cdot \boldsymbol{n}}{V^2}\left\{\boldsymbol{V}\wedge\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right)\right\}$$
$$-\left(\frac{1-\beta}{V^2}\right)\left\{\boldsymbol{n} \cdot \boldsymbol{V}\wedge\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right)\right\}\boldsymbol{V}$$
$$= \boldsymbol{n}\wedge\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right)-\{(1-\beta)+\beta(1-\beta)\}\left(\frac{\boldsymbol{n} \cdot \boldsymbol{V}}{V^2}\right)\left\{\boldsymbol{V}\wedge\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right)\right\}$$
$$= \left\{\boldsymbol{n}+\beta^2\left(\frac{\boldsymbol{n} \cdot \boldsymbol{V}}{c}\right)\frac{\boldsymbol{V}}{c}\right\}\wedge\left(\boldsymbol{E}+\frac{\boldsymbol{V}}{c}\wedge\boldsymbol{B}\right).$$

We have therefore shown that the general expressions which we obtained earlier for the force and couple arise by Lorentz transformation from the static expressions for the force and couple. In seeking these expressions we have assumed the vector and six-vector character of force and couple (also moment), but no other general assumptions have been made, so that the formulae for the force and couple in this case could have been deduced without the need of special methods.

Appendix

We derive expression (2.4) for the modulus of $\boldsymbol{n}$. From $q^\alpha q_\alpha = -M^2$, $q^\alpha V_\alpha = 0$ we have

$$q^2 = M^2 + \boldsymbol{q}^2, \qquad q = \frac{\boldsymbol{q}\,.\,\boldsymbol{V}}{c}.$$

Thus using

$$q = \beta^2\left(\frac{\boldsymbol{n}\,.\,\boldsymbol{V}}{c}\right)$$

$$\begin{aligned}\boldsymbol{n}^2 &= \left(\boldsymbol{q}-\frac{q\boldsymbol{V}}{c}\right)^2\\ &= (M^2+q^2)-2q^2+\frac{q^2V^2}{c^2}\\ &= M^2-\frac{q^2}{\beta^2}\\ &= M^2-\beta^2\left(\frac{\boldsymbol{n}\,.\,\boldsymbol{V}}{c}\right)^2.\end{aligned}$$

Writing $\boldsymbol{n} = |\boldsymbol{n}|\hat{\boldsymbol{n}}$ we obtain

$$\begin{aligned}|\boldsymbol{n}|^2 &= M^2\left\{1+\beta^2\left(\frac{\hat{\boldsymbol{n}}\,.\,\boldsymbol{V}}{c}\right)^2\right\}^{-1}\\ &= M^2\left(1-\frac{V^2}{c^2}\right)\left\{1-\frac{V^2}{c^2}+\left(\frac{\hat{\boldsymbol{n}}\,.\,\boldsymbol{V}}{c}\right)^2\right\}^{-1}\end{aligned}$$

hence

$$|\boldsymbol{n}| = M\left(1-\frac{V^2}{c^2}\right)^{1/2}\left(1-\frac{|\hat{\boldsymbol{n}}\wedge\boldsymbol{V}|^2}{c^2}\right)^{-1/2}.$$

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